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Some Fourier–Stieltjes Transforms of Absolute Value One

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WITH ADMIRATION AND RESPECT TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

1. INTRODUCTION

In this note, all notation and terminology not otherwise explained are as in the treatise [3]. Let G be a locally compact Abelian group, written multiplicatively and with identity element e. Let X be the character group of G. Let $\mathbf{M}(G)$ be the [convolution] measure algebra of G. Let ϵ_a be the Dirac measure assigning the mass 1 to the point $a \in G$. For $\mu \in \mathbf{M}(G)$, let $\hat{\mu}$ denote the Fourier–Stieltjes transform of μ, i.e., the function on X defined by

$$\hat{\mu}(\chi) = \int_G \overline{\chi(t)}\, d\mu(t) \qquad \text{for} \quad \chi \in X. \tag{1}$$

G. L. Seever has asked the writer whether or not every G with more than one element admits a measure μ, not of the form $\alpha\epsilon_a$ with $|\alpha| = 1$, for which we have

$$|\hat{\mu}(\chi)| = 1 \qquad \text{for all} \quad \chi \in X. \tag{2}$$

The question of finding such measures has a long history, and we here offer a modest contribution by constructing a certain class of purely discontinuous measures with property (1).

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2. Some Generalities

For $\mu \in \mathbf{M}(G)$, write $\tilde{\mu}$ for the measure adjoint to μ: $\tilde{\mu}(E) = \overline{\mu(E^{-1})}$ for all [say] Borel sets $E \subset G$. The identity (2) is clearly equivalent to the convolution identity

$$\mu * \tilde{\mu} = \epsilon_e, \tag{3}$$

or

$$\mu^{-1} = \tilde{\mu}. \tag{4}$$

This follows at once from the uniqueness theorem for Fourier–Stieltjes transforms (see, e.g., [3, (31.5)]) and the identity

$$\hat{\tilde{\mu}} = \hat{\mu}. \tag{5}$$

Let $\mu \in \mathbf{M}(G)$ be such that $\tilde{\mu} = \mu$. Define $\exp(i\mu)$ as usual by

$$\exp(i\mu) = \sum_{k=0}^{\infty} \frac{1}{k!} (i\mu)^k. \tag{6}$$

It is obvious that

$$(\exp(i\mu))\hat{} = \exp(i\hat{\mu}) \tag{7}$$

and so in view of (5), $\exp(i\mu)$ satisfies (2).

K. R. Stromberg [letter to the writer] has given another construction. Suppose that $\tilde{\mu} = \mu$ and that $\|\mu^2\| \leq 1$. Define

$$\nu = \sum_{k=0}^{\infty} \binom{1/2}{k} (-1)^k \mu^{2k}. \tag{8}$$

It is easy to check that $\nu^2 = \epsilon_e - \mu^2$ (see, e.g., [3, (21.19)]), and so

$$\epsilon_e = (\nu + i\mu)(\nu - i\mu) = (\nu + i\mu)(\nu + i\mu)^{\sim}. \tag{9}$$

That is,

$$\nu + i\mu \tag{10}$$

satisfies (3).

Glicksberg [1], in giving a very short proof of a theorem of Beurling and Helson [2], has pointed out that if μ satisfies (3), then the discontinuous part μ_d of μ also satisfies (3). For nonzero continuous μ, Stromberg's measure (10) obviously has a nonzero continuous part.

In his monograph [4], J. L. Taylor has classified the invertible elements in $\mathbf{M}(G)$ [4, Theorem 8.2.4]. For the case $G = \mathbf{R}$, Taylor [private communication] has provided the following classification of all $\mu \in \mathbf{M}(\mathbf{R})$ such that $\tilde{\mu} = \mu^{-1}$. Let η be the measure $f - \epsilon_0$, where f denotes the absolutely continuous measure with Radon–Nikodym derivative $f(x) = 2(2\pi)^{1/2} e^{-x}$

for $x \leqslant 0$, $f(x) = 0$ for $x > 0$. [The transform of this measure is $(1 + it)(1 - it)^{-1}$.] Let k be an integer, and δ any measure in $\mathbf{M}(\mathbf{R})$ such that $\tilde{\delta} = \delta$. The solutions of (4) in $\mathbf{M}(\mathbf{R})$ are exactly the measures

$$\eta^k * \epsilon_a * \exp(i\delta), \tag{11}$$

a being any element of $\mathbf{R}$. Quite probably [4, Theorem 8.2.4] can likewise be used to find all solutions of (4) for an arbitrary locally compact Abelian group G.

3. Certain Solutions of (4)

We carry out a specific construction of solutions of (4) in several steps. We look at non-Abelian G as well.

(3.1) Let G be finite and Abelian. Then $\mathbf{M}(G)^\wedge$ consists of all complex-valued functions on X, the group X being of course isomorphic to G. The functions ϕ on X of the form $(\alpha\epsilon_a)^\wedge$ for $|\alpha| = 1$ and $a \in G$ are exactly the functions of absolute value 1 on X such that $\phi/\phi(1)$ is a character of X. As there are just card(X) characters of X, there are evidently $\mathfrak{c}$ solutions of (4) on G not of the form $\alpha\epsilon_a$.

(3.2) Let G be finite and non-Abelian, with [finite] dual object Σ. Solutions of (4) on G are exactly the functions f on G such that $\hat{f}(\sigma)$ is a unitary operator on H_σ for all $\sigma \in \Sigma$. [For the notation used here, see [3, §27]). For the function [or measure] $\alpha\epsilon_a$, the Fourier transform at $\sigma \in \Sigma$ is the particular operator $\alpha U_a^{(\sigma)}$. For each σ, there are only card(G) operators $U_a^{(\sigma)}$, and so there are $\mathfrak{c}$ functions on G not of the form $\alpha\epsilon_a$ that are solutions of (4).

(3.3) Now take G to be the additive group $\mathbf{Z}$ of all integers, with the discrete topology. The character group $\mathbf{T}$ of $\mathbf{Z}$ is realized as the multiplicative group of all complex numbers of absolute value 1. We now choose: a number $p \in]1, 2]$; a number γ in $]0, 1[$; a real-valued absolutely continuous function f on $[-\pi, \pi]$ such that $|f| \leqslant \gamma, f(-\pi) = f(\pi)$, and f' is in $\mathfrak{L}_p(-\pi, \pi)$. Let g be the function $(1 - f^2)^{1/2}$ (nonnegative square root). Finally, define the function ϕ as $f + ig$.

It is clear that ϕ is absolutely continuous and that $\phi' \in \mathfrak{L}_p(-\pi, \pi)$. Regard ϕ as a function on $\mathbf{T}$, $\phi(e^{it}) = \phi(t)$ for $-\pi \leqslant t < \pi$. For $k \in \mathbf{Z}$, $k \neq 0$, we have

$$\begin{aligned} \hat{\phi}(k) &= \int_{\mathbf{T}} (e^{it})^{-k} \phi(e^{it})\, d\lambda(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt}\phi(t)\, dt \\ &= \frac{1}{2\pi ik} \int_{-\pi}^{\pi} e^{-ikt}\phi'(t)\, dt = \frac{1}{ik} (\phi')^\wedge (k). \end{aligned} \tag{12}$$

Write q for the number $p/(p-1)$. The Hausdorff-Young inequality and (12) imply that

$$\begin{aligned}\sum_{k\in\mathbf{Z}} |\hat{\phi}(k)| &= |\hat{\phi}(0)| + \sum{}' |\hat{\phi}(k)| = |\hat{\phi}(0)| + \sum{}' \frac{1}{|k|} |(\phi')^\wedge(k)| \\ &\leq |\hat{\phi}(0)| + \left(\sum{}' |k|^{-p}\right)^{1/p} \left(\sum{}' |(\phi')^\wedge(k)|^q\right)^{1/q} \\ &\leq |\hat{\phi}(0)| + A \cdot \left(\int_{\mathbf{T}} |\phi'(e^{it})|^p\, dt\right)^{1/p} < \infty. \qquad (13)\end{aligned}$$

From (13), we see that ϕ is the sum of its own Fourier series:

$$\phi(e^{it}) = \sum_{k\in\mathbf{Z}} \hat{\phi}(k)\, e^{ikt}. \qquad (14)$$

Let μ be the measure on $\mathbf{Z}$ [i.e., the element of $l_1(\mathbf{Z})$] that assigns the mass $\hat{\phi}(-k)$ to the point k for all $k \in \mathbf{Z}$. Then $\hat{\mu}$ is the function ϕ on $\mathbf{T}$.

From our construction of ϕ, it is clear that the range $\phi(\mathbf{T})$ of ϕ can be any closed arc of length less than π lying in the upper half of $\mathbf{T}$. By considering $\phi(e^{it}e^{i\beta})$, we can position this arc wherever we like, and by taking ϕ^3 for example [which also has absolutely convergent Fourier series], we can obtain any arc we wish as $\phi(\mathbf{T})$.

For $\mathbf{M}(\mathbf{Z})$, our final result is the following. Let C be any closed arc contained in $\mathbf{T}$ [$C = \mathbf{T}$ is not excluded]. There exists a measure $\mu \in \mathbf{M}(\mathbf{Z})$ such that $\hat{\mu}(T) = C$.

(3.4) Now let G be any locally compact group containing an element a of infinite order. The closure H of the cyclic subgroup $\{a^k: k \in \mathbf{Z}\}$ is either compact or topologically isomorphic with $\mathbf{Z}$ (see e.g., [3, (9.1)]). Note that H can be any monothetic compact group. Let μ be any measure on $\mathbf{Z}$ as in (3.3) and let μ_H be the measure on G such that $\mu_H(\{a^k\}) = \mu(\{k\})$ for all $k \in \mathbf{Z}$. It is easy to check that $(\mu_H)^\sim = (\mu_H)^{-1}$ [in the algebra $\mathbf{M}(G)$]. Thus we obtain a large class of solutions of the equation (3). For the case of Abelian G, a few comments about the range $(\mu_H)^\wedge(X)$ may be in order. This set consists of all the numbers $\sum_{k\in Z} \mu(\{k\})\, e^{-ikt}$ for which the function $a^k \to e^{ikt}$ $(k \in \mathbf{Z})$ has a continuous extension over H. The e^{it} for which this holds can be any infinite subgroup W of $\mathbf{T}$ (see [3, (25.12)]). Thus the range $(\mu_H)^\wedge(X)$ is the set $\hat{\mu}(W)$, which is dense in the arc $\hat{\mu}(\mathbf{T})$, but seems difficult to describe completely.

(3.5) Finally let G be a locally compact torsion group. Finite G's are dealt with in (3.1) and (3.2) above. If G contains an infinite Abelian subgroup, then G contains an infinite ascending chain of finite Abelian subgroups. On each of these, construct a measure as in (3.1). A weak limit point of these measures can be found which will provide nontrivial solutions of (3). An alternative construction is to repeat (3.3) for $\mathbf{Z}(p^\infty)$, and direct sums of countably many cyclic groups of finite order. This could be done, and we leave the details to any interested reader. If G contains no infinite Abelian subgroup [already something of a rarity], then we can only use (3.2), without going far afield into the algebraic theory of groups.

References

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